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From the von Neumann Equation to the Quantum Boltzmann Equation II: Identifying the Born Series

F. Castella[1]

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In a previous paper [Ca1], the author studied a low density limit in the periodic von Neumann equation with potential, modified by a damping term. The model studied in [Ca1], considered in dimensions $d \geqslant 3$, is deterministic. It describes the quantum dynamics of an electron in a periodic box (actually on a torus) containing one obstacle, when the electron additionally interacts with, say, an external bath of photons. The periodicity condition may be replaced by a Dirichlet boundary condition as well. In the appropriate low density asymptotics, followed by the limit where the damping vanishes, the author proved in [Ca1] that the above system is described in the limit by a linear, space homogeneous, Boltzmann equation, with a cross-section given as an explicit power series expansion in the potential. The present paper continues the above study in that it identifies the cross-section previously obtained in [Ca1] as the usual Born series of quantum scattering theory, which is the physically expected result. Hence we establish that a von Neumann equation converges, in the appropriate low density scaling, towards a linear Boltzmann equation with cross-section given by the *full* Born series expansion: we do not restrict ourselves to a weak coupling limit, where only the first term of the Born series would be obtained (Fermi's Golden Rule).

KEY WORDS: Density matrix; low density limit; time-dependent scattering theory; Fermi's Golden Rule; oscillatory integrals.

1. INTRODUCTION

The present paper is the continuation of a previous work [Ca1] of the author (see also the announcement of a weaker result [CD]). In [Ca1], it is proved that a von Neumann equation with a damping term converges along some low density limit towards a linear, space homogeneous, Boltzmann

[1] CNRS et IRMAR, Université de Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France; e-mail: castella@maths.univ-rennes1.fr

equation. We recall below the precise asymptotics and convergence results obtained in [Ca1] (Theorem 1 below). However, the cross-section exhibited in [Ca1] has a rather complicated expression (see (2.11)), and its connection with the usual Born series is by no means clear. In this context, the present paper identifies the cross-section appearing in [Ca1] as the usual Born series of quantum scattering. It is important to note that the present paper obtains, along some low density asymptotics in a von Neumann equation, a Boltzmann equation with cross-section given by the *full* Born series. In the simpler case of a weak coupling asymptotics, the resulting cross-section would reduce to the *first* term of the Born series expansion. This point also answers questions raised at the physical level in [Co], see point c) of the introduction below.

1.1. The Physical Context

Let us now give the context in which the present paper takes place. The general question is the following: let us consider the quantum evolution of an electron (or a beam of non-interacting electrons) in a field of obstacles. This situation is *a priori* described by the Schrödinger equation, or more generally the von Neumann equation, where a perturbing potential describes the interaction of the electron with the obstacles. To simplify things, one may assume that each obstacle, labelled by the index $j \in \mathbb{N}$ and centered around the position $X_j \in \mathbb{R}^d$ (throughout the paper we shall assume $d \geqslant 3$), creates the potential $\lambda V(\mathbf{x}-X_j)$ at $\mathbf{x} \in \mathbb{R}^d$, where V is a fixed, real-valued profile, and $\lambda \in \mathbb{R}$ is a coupling constant. Here and in the sequel, we always assume that V is small, smooth, and decaying enough so that a reasonable scattering theory is at hand for the potential V. Hence the electron undergoes the influence of the total potential,

$$V_{\text{tot}}(\mathbf{x}) = \lambda \sum_j V(\mathbf{x}-X_j) \tag{1.1}$$

at $\mathbf{x}$, where the sum is locally finite (say). Physically, such a situation is expected to describe the evolution of an electron in a distribution of impurities, and a typical application of such a model is the analysis of semi-conductor devices (see [MRS], or also [Fi]).

Now, the solution of the Schrödinger, or the von Neumann equation, with potential V_{tot} is often too complicated and one looks for asymptotic models. The typical regime under interest considers large times and small values of the potential: in such regimes, it is physically expected (see [Pa], [VH1,2,3], [KL1,2], [Ku], [Pr], [Vk], [Zw], or also [Ck], see [Fi] for recent developments, see also [KPR] for a more mathematical but still formal approach) that the dynamics of the electron may be asymptotically described by a linear Boltzmann equation of the form,

$$\partial_t f(t, \mathbf{n}) = \int_{\mathbb{R}^d} [\Sigma(\mathbf{n}, \mathbf{k}) f(t, \mathbf{k}) - \Sigma(\mathbf{k}, \mathbf{n}) f(t, \mathbf{n})] \, d\mathbf{k} \tag{1.2}$$

in the space homogeneous case, or more generally,

$$\partial_t f(t, \mathbf{x}, \mathbf{n}) + \mathbf{n} \cdot \nabla_{\mathbf{x}} f(t, \mathbf{x}, \mathbf{n}) = \int_{\mathbb{R}^d} [\Sigma(\mathbf{n}, \mathbf{k}) f(t, \mathbf{x}, \mathbf{k}) - \Sigma(\mathbf{k}, \mathbf{n}) f(t, \mathbf{x}, \mathbf{n})] \, d\mathbf{k} \tag{1.3}$$

in the space inhomogeneous case. Here, $f(t, \mathbf{n})$ (respectively $f(t, \mathbf{x}, \mathbf{n})$) represents the probability at time $t \in \mathbb{R}$ that the electron has the momentum $\mathbf{n} \in \mathbb{R}^d$ (and possibly position $\mathbf{x} \in \mathbb{R}^d$). Also, the right-hand-side of (1.2) (as well as (1.3)) has the usual structure of a gain term plus a loss term: at time t (and possibly position $\mathbf{x}$), the electron may jump from the momentum $\mathbf{k}$ to the momentum $\mathbf{n}$, with probability $\Sigma(\mathbf{n}, \mathbf{k})$, hence the contribution $\int \Sigma(\mathbf{n}, \mathbf{k}) f(t, \mathbf{k}) \, d\mathbf{k}$ in (1.2)—this is the gain term—but it may symmetrically jump from the momentum $\mathbf{n}$ to another momentum $\mathbf{k}$, with the probability $\Sigma(\mathbf{k}, \mathbf{n})$, hence the contribution $-\int \Sigma(\mathbf{k}, \mathbf{n}) f(t, \mathbf{n}) \, d\mathbf{n}$ in (1.2)—this is the loss term. The quantity $\Sigma(\mathbf{n}, \mathbf{k})$ is usually called the cross-section.

The physically relevant value of the cross-section Σ depends on the exact asymptotic regime considered in the original Schrödinger, or von Neumann equation. One distinguishes two main regimes. In the *weak coupling* limit (also known as the Van Hove limit), the obstacles are distributed so that one typically finds one obstacle per unit volume, but the coupling constant λ in (1.1) is small, and long times of the order $1/\lambda^2$ are considered. The mathematically relevant limit is $\lambda \to 0$. Since each encounter with an obstacle has an effect of the order of magnitude λ^2 on the dynamics (this is a consequence of the Fermi Golden Rule (1.7) below), the weak coupling regime corresponds to a case where the electron typically undergoes many "collisions" with the obstacles per unit time in the new time scale (typically $1/\lambda^2$), but each "collision" affects the electron by a small quantity of the order λ^2, so that the total effect of the obstacles on the dynamics of the electron is of the order 1. The second regime is the *low density* regime, also known under the name of Boltzmann–Grad limit. Here obstacles are distributed so that one finds a small amount ε ($\varepsilon \to 0$) of obstacles per unit volume, and long time scales of the order $1/\varepsilon$ are considered. Also, the coupling constant λ is kept of the order 1. In this regime, the electron typically meets one obstacle per unit time in the new time scale, but each encounter with an obstacle has immediately an effect of the order 1 on the dynamics.

These two different regimes are expected to give two different cross-sections. In the low density regime, one expects that the dynamics of the electron is indeed asymptotically described by an equation of the form (1.2) (or (1.3)), the relevant cross-section Σ satisfying in this case $\Sigma = \Sigma^{\mathrm{ld}}$ where,

$$\Sigma^{\mathrm{ld}}(\mathbf{n}, \mathbf{k}) = 2\pi\delta(\mathbf{n}^2 - \mathbf{k}^2)\, |T(\mathbf{k}, \mathbf{n})|^2 \tag{1.4}$$

Here T is the usual T-matrix of quantum theory (see [RS]), naturally associated with the potential λV, and expressed in the momentum representation. It is defined as,

$$S(\mathbf{n}, \mathbf{k}) = \delta(\mathbf{n} - \mathbf{k}) - 2i\pi T(\mathbf{n}, \mathbf{k}) \tag{1.5}$$

where S is the scattering operator associated with λV (again in the momentum representation). It is known [RS] that $|T|^2$ admits a power series expansion in terms of the potential λV, called the Born series, whose first term is given by,

$$|T(\mathbf{n}, \mathbf{k})|^2 = \lambda^2\, |\hat{V}(\mathbf{n} - \mathbf{k})|^2 + O(\lambda^3) \tag{1.6}$$

where $\hat{V}$ is the Fourier transform of the potential V. In the weak coupling regime on the other hand, one also expects that the dynamics of the electron is asymptotically described by an equation of the form (1.2) (or (1.3)), the relevant cross-section Σ satisfying in this case $\Sigma = \Sigma^{\mathrm{wc}}$ where,

$$\Sigma^{\mathrm{wc}}(\mathbf{n}, \mathbf{k}) = 2\pi\delta(\mathbf{n}^2 - \mathbf{k}^2)\, |\hat{V}(\mathbf{n} - \mathbf{k})|^2 \tag{1.7}$$

This equality is known under the name of "Fermi Golden Rule". Such cross-sections are routinely considered in the modelling of semi-conductor devices ([MRS], [Fi]). We emphasize in passing that Σ^{wc} somehow corresponds to the lower order expansion of Σ^{ld} in the potential.

1.2. Mathematical Derivations

Numerous mathematical works have rigorously studied the convergence of the Schrödinger equation towards equations of the form (1.2) or (1.3) *in the weak coupling limit*, and the cross-section (1.7) is indeed derived. We wish to quote the stochastic approach developped in [Sp], [HLW], [La], [EY1,2] (see also the more recent work [PV], where the authors consider a different stochastic framework). All these works consider the case of randomly distributed obstacles ($X_j \equiv X_j(\omega)$, where ω is the stochastic parameter and the X_j's are as in (1.1)), and the convergence holds in expectation with respect to ω, or almost surely. Obviously, one key difficulty in such a rigorous derivation lies in the fact that the original time-reversible Schrödinger equation is expected to converge towards the time-irreversible Boltzmann equation, which justifies that the convergence can by no means hold in any "strong" sense (e.g. without removing some exceptional zero-measure set).

For a different approach, handling the *low density limit*, we also wish to quote [Dü]: here the author considers the quantum dynamics of an atom (or a system with a finite number of energy levels), coupled with a Fermi gas of electrons at thermal equilibrium. The reduced dynamics of the atom is proved to be asymptotically described by a quantum dynamical semi-group involving a linear Boltzmann operator, with cross-section given by the appropriate Born series expansion. In the same spirit, but for different kind of limiting equations, let us mention the work [CEFM], where the dynamics of an electron coupled to a system of harmonic oscillators is proved to converge, in the appropriate scaling limit, towards a Fokker–Planck equation.

Related works in a deterministic framework are [Ni1,2], or also [Ca3], but these works do not give the convergence towards a true Boltzmann equation of the form (1.2) nor (1.3).

1.3. The Present Model: Description of the Regime, Results Obtained in [Ca1]

The above mentioned works [Sp], [HLW], [La], [EY1,2] thus assert that, outside some "exceptional configurations" of the obstacles, the convergence of the associated Schrödinger equation towards a Boltzmann equation indeed holds. It is natural to ask whether, for *one particular, deterministic*, configuration of the obstacles, the same convergence holds.

In this context, the first key motivation for the present work together with [Ca1] is to study the convergence of the von Neumann equation towards the linear Boltzmann equation (1.2) in one particular situation, namely the periodic one.

As it is clear below, a second strong motivation is to give a rigorous basis to the physical approach of conventional scattering theory (see e.g. [CTDL]): here, computations are usually performed on systems with discrete spectrum (e.g. the Laplacian in a finite box), and the size of the box is eventually set to infinity to recover systems with continuous spectrum. This procedure of "taking the size of the box to infinity" is questioned from a physical point of view in [Co], and the present paper actually gives a mathematical answer to the questions raised in [Co]. We readily mention that all the results stated here and below in the periodic case hold as well when the periodic boundary conditions are replaced by

Dirichlet boundary conditions, see [Ca1] (the periodic framework is chosen only for notational convenience: the eigenfunctions $\exp(i\,\mathbf{nx})$ arising in the periodic case simply have to be replaced by $\cos(\mathbf{nx})$ in the case of Dirichlet boundary conditions, hence the need for extra, but unimportant, symmetrisations in the latter case).

The model is the following: as in conventional scattering theory (see [Ck], [CTDL], [Boh], see also [Co]), one considers an electron in a large periodic box of size L, $[-\pi L, \pi L]^d \subset \mathbb{R}^d$, with periodic boundary conditions. One smooth potential with compact support of size 1 is set at the origin, so that the density of obstacles readily is of the order $1/L^d$. Since we wish to consider a low density limit, times of the order L^d are considered in [Ca1]. In the Fourier space (indexed by integer numbers $n \in \mathbb{Z}^d$, since the original model is posed in a box of finite volume), the usual von Neumann equation describing the dynamics of the electron is,

$$\begin{aligned}\frac{i}{L^d}\,\partial_t \rho^L(t, n, p) = {} & \frac{p^2 - n^2}{L^2}\,\rho^L(t, n, p)\\ & + \frac{\lambda}{L^d} \sum_{k \in \mathbb{Z}^d} \left[\hat{V}\left(\frac{n-k}{L}\right) \rho^L(t, k, p) - \hat{V}\left(\frac{k-p}{L}\right) \rho^L(t, n, k)\right]\end{aligned} \tag{1.8}$$

(one recognizes the Fourier transform of usual commutator with $-\Delta_{\mathbf{x}} + \lambda V(\mathbf{x})$ on the right-hand-side of (1.8)—see [Ca1] for details on the normalizations). The initial datum in (1.8) is taken of the form,

$$\rho^L(0, n, p) = \frac{1}{L^d}\,\rho^0\left(\frac{n}{L}\right) \mathbf{1}[n = p] \tag{1.9}$$

where ρ^0 is a given, smooth and decaying profile. Here, $\rho^L(t, n, p)$ is the so-called density matrix of the electron, indexed by the scaling parameter L: the diagonal values $\rho^L(t, n, n)$ represent the probability, at time t, that the electron is in the eigenstate $(2\pi L)^{-d/2} \exp(in \cdot \mathbf{x}/L)$ of the Laplacian $-\Delta_{\mathbf{x}}$ in the periodic box $[-\pi L, \pi L]^d$, and the off-diagonal values $\rho^L(t, n, p)$ $(n \neq p)$ represent correlations between the various occupation numbers $\rho^L(t, n, n)$. The initial datum (1.9) is a generalization of the usual thermodynamical equilibrium with inverse temperature β for the free von Neumann equation (i.e. Eq. (1.8) for $\hat{V} \equiv 0$), for which $\rho^0(\mathbf{n}) = \exp(-\beta \mathbf{n}^2)$. Also, $\hat{V}$ represents the usual Fourier transform of the potential V, defined as,

$$\hat{V}(\mathbf{n}) = \int_{\mathbb{R}^d} V(\mathbf{x}) \exp(-i\,\mathbf{n} \cdot \mathbf{x})\, d\mathbf{x} \qquad \left(= \int_{[-\pi L, \pi L]^d} V(\mathbf{x}) \exp(-i\,\mathbf{n} \cdot \mathbf{x})\, d\mathbf{x}\right) \tag{1.10}$$

and the last equality comes from the assumption of compact support on V. The low density limit in (1.8) corresponds to the limit $L \to \infty$ in (1.8).

This is where some care has to be taken. Equation (1.8) describes the quantum evolution of an electron on a torus. This is a highly specific, as well as a non-generic case. It is actually proved in [CP1, 2] that the low density limit $L \to \infty$ in (1.8) *does not* give the desired Boltzmann equation (1.2) with cross-section (1.4): mathematically speaking, one needs some extra, regularizing parameter, and this turns out to be a physical necessity as well, see [Hu]. Indeed, as readily seen on (1.8), the periodicity gives rise to specific phase coherence effects: roughly speaking, the solution of (1.8) gives rise to highly oscillating factors $\exp(iL^{d-2}[n^2 - p^2]\, t)$, so that the contribution of the set of integer numbers n and p such that $n^2 = p^2$ turns out to abnormally dominate the asymptotic process. In [CP1, 2], this effect is precisely quantified in terms of arithmetic considerations, and the main result is that the asymptotic dynamics $L \to \infty$ in (1.8) remains time-reversible. We mention in passing that the above mentioned non-convergence result [CP1, 2] in the fully periodic case is somehow not surprising: the low density limit for a *classical* particle moving through a *periodic* distribution of hard spheres *does not* converge towards the physically expected Boltzmann equation neither, as proved in [BGW], contrary to the case of a classical particle moving through random obstacles, treated in [BBS].

For all these reasons, the exact model in which the low density limit is performed in [Ca1] is a modified version of (1.8). To be specific, we modify our model so as to take into account an additional interaction of the electron with, typically, an external bath of phonons, at least at a phenomenological level. Physically, such an interaction leads to an exponential decay of both the diagonal and the off-diagonal part of the density-matrix, but the decay in the off-diagonal part typically is much quicker than that of the diagonal part. Hence, following [NM], [SSL], [Boy], [Lo], this leads to the introduction of an additional damping term in (1.8), measured by the small damping parameter $\alpha > 0$, and acting on the off-diagonal part of the density matrix only. Thus, in [Ca1], we start with,

$$\begin{aligned} &\frac{i}{L^d}\, \partial_t \rho^{L,\alpha}(t, n, p) \\ &\quad = \frac{p^2 - n^2}{L^2}\, \rho^{L,\alpha}(t, n, p) - i\alpha \rho^{L,\alpha}(t, n, p)\, \mathbf{1}[n \neq p] \\ &\qquad + \frac{\lambda}{L^d} \sum_{k \in \mathbb{Z}^d} \left[\hat{V}\left(\frac{n-k}{L}\right) \rho^{L,\alpha}(t, k, p) - \hat{V}\left(\frac{k-p}{L}\right) \rho^{L,\alpha}(t, n, k) \right] \end{aligned} \tag{1.11}$$

with initial datum given by (1.9) as well. Here, we emphasize the dependence of the density matrix upon the two scaling parameters L and α. Note that the introduction of a damping term readily makes the original model (1.11) in which we pass to the limit *time irreversible*. We mention in passing that the additional damping term involved in (1.11) satisfies the so-called Linblad-property ([Li]), as proved in [Ca1].

Associated with the sequence of occupation numbers $\rho^{L,\alpha}(t, n, n)$, we build up the natural distribution,

$$f^{L,\alpha}(t, \mathbf{n}) := \sum_{n \in \mathbb{Z}^d} \rho^{L,\alpha}(t, n, n)\, \delta\left(\mathbf{n} - \frac{n}{L}\right) \tag{1.12}$$

The main theorem obtained in [Ca1] asserts the following (we refer to [Ca1] for more precise and complete statements),

Theorem 1 ([Ca1]). Let $f^{L,\alpha}$ be as in (1.12), where $\rho^{L,\alpha}$ solves (1.11) with initial datum ρ^0. Let $D \geqslant d+1$ and assume the initial datum ρ^0 and the potential V have the following regularity,

$$\begin{aligned} \|\rho^0\|_{\mathcal{T}_D(\mathbb{R}^d)} &:= \|(1+\mathbf{n}^2)^{D/2}\, \rho^0(\mathbf{n})\|_{L^\infty(\mathbb{R}^d)} < \infty \\ \|\hat{V}\|_{\mathcal{S}_{2D}(\mathbb{R}^d)} &:= \sum_{|\gamma| \leqslant 2D} \|(1+\mathbf{n}^2)^D\, \partial_n^\gamma \hat{V}(\mathbf{n})\|_{L^\infty(\mathbb{R}^d)} < \infty \end{aligned} \tag{1.13}$$

Assume also that $|\lambda| \leqslant \lambda_0$ for some small enough $\lambda_0 > 0$ whose value only depends upon the norms of ρ^0 and $\hat{V}$ in the above spaces. Assume finally that $d \geqslant 3$. Then,

(i) the following non-commuting limit exists in $C^0(\mathbb{R}_t^+; [\mathcal{S}_{2D}(\mathbb{R}^d)]^* -$ weak*), as well as $[L^1(\mathbb{R}_t^+;\ \mathcal{T}_D(\mathbb{R}^d))]^*$ $-$weak*, where E^* denotes the dual space of the Banach space E,

$$f(t, \mathbf{n}) = \lim_{\alpha \to 0}\ \lim_{L \to \infty}\ f^{L,\alpha}(t, \mathbf{n}) \tag{1.14}$$

(ii) f satisfies in the distribution sense an equation of the form,

$$\begin{aligned} \partial_t f(t, \mathbf{n}) &= \int_{\mathbb{R}^d} [\Sigma^1(\mathbf{n}, \mathbf{k})\, f(t, \mathbf{k}) - \Sigma^2(\mathbf{n}, \mathbf{k})\, f(t, \mathbf{n})]\, d\mathbf{k} \\ f(0, \mathbf{n}) &= \rho^0(\mathbf{n}) \end{aligned} \tag{1.15}$$

for some cross-sections Σ^1 and Σ^2 whose value is given as an explicit power series expansion in λ (see (2.7) below for the explicit formulae).

1.4. Statement of Our Main Theorem

In this context, the main theorem of the present paper is the following,

Theorem 2. Under the assumptions of Theorem 1 above, the cross-sections Σ^1 and Σ^2 appearing in Eq. (1.15), as derived in [Ca1] (see (2.7) below) coincide with the low density cross-section Σ^{ld} defined above see (1.4)), in the sense that,

$$\begin{gathered}\Sigma^1(\mathbf{n},\mathbf{k}) = \Sigma^{\mathrm{ld}}(\mathbf{n},\mathbf{k}) = 2\pi\delta(\mathbf{n}^2-\mathbf{k}^2)\,|T(\mathbf{k},\mathbf{n})|^2 \\ \int_{\mathbb{R}^d} \Sigma^2(\mathbf{n},\mathbf{k})\,d\mathbf{k} = \int_{\mathbb{R}^d} \Sigma^{\mathrm{ld}}(\mathbf{k},\mathbf{n})\,d\mathbf{k} = 2\pi \int_{\mathbb{R}^d} \delta(\mathbf{n}^2-\mathbf{k}^2)\,|T(\mathbf{n},\mathbf{k})|^2\,d\mathbf{k}\end{gathered} \tag{1.16}$$

where T is the T-matrix associated with λV in the momentum representation.

The remainder part of the present paper is dedicated to the proof of the main theorem.

The interested reader may find in [Ca2] a review about the present work together with [Ca1], as well as about the non-convergence result obtained in [CP1, 2].

2. PROOF OF THE MAIN THEOREM

The proof is divided into two steps. **First** we recall the explicit value of the Born series expansion of the T-matrix associated with λV, and also recall the explicit value of the cross-sections Σ^1 and Σ^2 as derived in [Ca1] (Property 1). We introduce in passing the notations used throughout the paper (Defnition 1). The idea is that both the T-matrix and the two cross-sections Σ^1 and Σ^2 involved in (1.15) are naturally given as power series in λ (or, equivalently, λV). **Secondly**, we prove in several steps that the above mentioned series coincide *term by term* (Lemma 3). This identification relies on two facts: on the one hand, the power series in λ which define both Σ^1 and Σ^2 can be built up using a simple iteration procedure (see (1.24)–(1.25) and Lemma 2). On the other hand, the implementation of this iteration procedure is greatly simplified upon the use of a Lemma (Lemma 1) identifying the sum of certain oscillatory integrals naturally arising in the formulation of the problem.

2.1. Explicit value of the Born series expansion, the cross-sections Σ^1 and Σ^2

We begin with some notations,

Definition 1. (i) Let $\hat{V}$ be the Fourier transform of V as defined in (1.10). Then, for any $\mathbf{n}, \mathbf{k}_1, \ldots, \mathbf{k}_l, \mathbf{p}$ in $\mathbb{R}^d$, we define,

$$\hat{V}_{l+1}(\mathbf{n}, \mathbf{k}_1, \ldots, \mathbf{k}_l, \mathbf{p}) := i^{l+1} \hat{V}(\mathbf{n}-\mathbf{k}_1)\, \hat{V}(\mathbf{k}_1-\mathbf{k}_2) \cdots \hat{V}(\mathbf{k}_{l-1}-\mathbf{k}_l)\, \hat{V}(\mathbf{k}_l-\mathbf{p}) \tag{2.1}$$

Note that, since V is real valued, and if * denotes complex conjugation, we have,

$$[\hat{V}_{l+1}(\mathbf{n}, \mathbf{k}_1, \ldots, \mathbf{k}_l, \mathbf{p})]^* = (-1)^{l+1}\, \hat{V}_{l+1}(\mathbf{p}, \mathbf{k}_l, \ldots, \mathbf{k}_1, \mathbf{n}) \tag{2.2}$$

(ii) We introduce the distribution over $\mathbb{R}^{2d}$,

$$\Delta(\mathbf{n}, \mathbf{p}) := \int_{s=0}^{+\infty} \exp(i[\mathbf{n}^2-\mathbf{p}^2]\, s)\, ds \tag{2.3}$$

It satisfies,

$$\Delta(\mathbf{n}, \mathbf{p}) = \pi\delta(\mathbf{n}^2-\mathbf{p}^2) + i\,\mathrm{pv}\left(\frac{1}{\mathbf{n}^2-\mathbf{p}^2}\right) \tag{2.4}$$

Remark 1. Upon using standard theorems about composition of distributions (See [Hö]), the Dirac mass and principal value involved in (2.4) are easily seen to be well-defined, at least in dimensions $d \geqslant 3$. The identity between the oscillatory integral on the left-hand-side of (2.4) and its right-hand-side is easily proved as well. However, as we will see below, the cross-sections Σ^1 and Σ^2 derived in [Ca1] naturally involve *products* of such distributions, and a typical product is of the form $\Delta(\mathbf{n}, \mathbf{k}_1)\, \Delta(\mathbf{n}, \mathbf{k}_2) \cdots \Delta(\mathbf{n}, \mathbf{k}_l)$, say, l being a large integer parameter. One key difficulty handled in [Ca1] is to prove that such products are indeed well-defined as distributions (this is *not* a consequence of the standard theorems about products of distributions having certain properties on their wave fronts), and to control the regularity of these products in, say, Sobolev spaces with negative exponents in terms of l: the exponent should not grow too fast with l. Both tasks are accomplished in [Ca1], upon considering these products as oscillatory integrals, and upon explicitely using the fact that the phase in (2.3) is quadratic. We give in Lemma 4 of the appendix a version of the needed regularity result that will be sufficient for our purposes. ∎

With these notations, we are able to formulate both T and Σ^1, Σ^2, as power series expansion in λ,

Proposition 1. Under the assumptions of Theorem 1, we have,

(i) The T-matrix associated with λV admits the following power series expansion,

$$T(\mathbf{n},\mathbf{k}) = \sum_{l\geqslant 0} \lambda^{l+1} T_l(\mathbf{n},\mathbf{k})$$

$$T_l(\mathbf{n},\mathbf{k}) := -i\int_{\mathbb{R}^{ld}} \Delta(\mathbf{k},\mathbf{k}_1)\,\Delta(\mathbf{k},\mathbf{k}_2)\cdots\Delta(\mathbf{k},\mathbf{k}_l) \times \hat{V}_{l+1}(\mathbf{n},\mathbf{k}_1,\ldots,\mathbf{k}_l,\mathbf{k})\,d\mathbf{k}_1\cdots d\mathbf{k}_l \tag{2.5}$$

with the obvious convention that $T_0(\mathbf{n},\mathbf{k}) := \hat{V}_1(\mathbf{n},\mathbf{k})$. This equality holds in the distribution sense.

(ii) As a consequence, the cross-section Σ^{ld} defined in (1.4) admits the power expansion, valid in the distribution sense,

$$\Sigma^{\mathrm{ld}}(\mathbf{n},\mathbf{k}) = 2\pi\delta(\mathbf{n}^2-\mathbf{k}^2)\sum_{l\geqslant 1}\lambda^{l+1}\Sigma^{\mathrm{ld}}_l(\mathbf{n},\mathbf{k})$$

$$\Sigma^{\mathrm{ld}}_l(\mathbf{n},\mathbf{k}) := \sum_{s=0}^{l-1}\int_{\mathbb{R}^{(l-1)d}}(-1)^{s+1}\,\Delta(\mathbf{k}_1,\mathbf{n})\cdots\Delta(\mathbf{k}_s,\mathbf{n})\,\Delta(\mathbf{n},\mathbf{k}_{s+1})\cdots\Delta(\mathbf{n},\mathbf{k}_{l-1}) \times\hat{V}_{l+1}(\mathbf{n},\mathbf{k}_1,\ldots,\mathbf{k}_s,\mathbf{k},\mathbf{k}_{s+1},\ldots,\mathbf{k}_{l-1},\mathbf{n})\,d\mathbf{k}_1\cdots d\mathbf{k}_{l-1} \tag{2.6}$$

Here, the convention is used that the integrand reduces to $\Delta(\mathbf{n},\mathbf{k}_1)\cdots\Delta(\mathbf{n},\mathbf{k}_l)\,\hat{V}_{l+1}(\mathbf{n},\mathbf{k}_1,\ldots,\mathbf{k}_{l-1},\mathbf{k},\mathbf{n})$ in the case $s=l-1$, and similarly if $s=0$.

(iii) The distribution $f=\lim_{\alpha\to 0^+}\lim_{L\to\infty} f^{L,\alpha}$ of Theorem 1 satisfies indeed an equation of the form (1.15), in that it satisfies in the distribution sense,

$$\partial_t f(t,\mathbf{n}) = \sum_{l\geqslant 1}\lambda^{l+1}\mathcal{Q}_l(f)(t,\mathbf{n})$$

$$\begin{aligned}\mathcal{Q}_l(f)(t,\mathbf{n}) := (2\,\mathrm{Re})\sum_{\varepsilon_1,\ldots,\varepsilon_l}(-1)^{\varepsilon_1+\cdots+\varepsilon_l}\int_{\mathbb{R}^{dl}}&\Delta(\mathbf{n}-\varepsilon_1\mathbf{k}_1,\mathbf{n}+\tilde{\varepsilon}_1\mathbf{k}_1)\\ &\times\Delta(\mathbf{n}-\varepsilon_1\mathbf{k}_1-\varepsilon_2\mathbf{k}_2,\mathbf{n}+\tilde{\varepsilon}_1\mathbf{k}_1+\tilde{\varepsilon}_2\mathbf{k}_2)\cdots\\ &\times\Delta(\mathbf{n}-\varepsilon_1\mathbf{k}_1-\cdots-\varepsilon_l\mathbf{k}_l,\mathbf{n}+\tilde{\varepsilon}_1\mathbf{k}_1+\cdots+\tilde{\varepsilon}_l\mathbf{k}_l)\\ &\times[i\hat{V}(\mathbf{k}_1)]\cdots[i\hat{V}(\mathbf{k}_l)][i\hat{V}^*(\mathbf{k}_1+\cdots+\mathbf{k}_l)]\\ &\times f(t,\mathbf{n}-\varepsilon_1\mathbf{k}_1-\cdots-\varepsilon_l\mathbf{k}_l)\,d\mathbf{k}_1\cdots d\mathbf{k}_l\end{aligned} \tag{2.7}$$

where the sum carries over all values of $(\varepsilon_1,\dots,\varepsilon_l)\in\{0,1\}^l$, and the convention $\tilde{\varepsilon}_i=1-\varepsilon_i$ is used.

(iv) Formulae (2.7) may be rewritten in the compact form,

$$\partial_t f(t,\mathbf{n})=-i\lambda\int_{\mathbb{R}^d}[\hat{V}(\mathbf{n}-\mathbf{k})\,g(t,\mathbf{k},\mathbf{n})-\hat{V}(\mathbf{k}-\mathbf{n})\,g(t,\mathbf{n},\mathbf{k})]\,d\mathbf{k} \quad (2.8)$$

up to introducing the auxiliary distribution,

$$\begin{aligned} g(t,\mathbf{n},\mathbf{p}) :=\; & -i\lambda\,\Delta(\mathbf{n},\mathbf{p})\,\hat{V}(\mathbf{n}-\mathbf{p})[f(t,\mathbf{p})-f(t,\mathbf{n})] \\ & -i\lambda\Delta(\mathbf{n},\mathbf{p})\int_{\mathbb{R}^d}[\hat{V}(\mathbf{n}-\mathbf{m})\,g(t,\mathbf{m},\mathbf{p})-\hat{V}(\mathbf{m}-\mathbf{p})\,g(t,\mathbf{n},\mathbf{m})]\,d\mathbf{m} \end{aligned} \quad (2.9)$$

Remark 2 (regularity). (i) It is well known that the Born series expansion (2.5) converges pointwise for λ small enough and V smooth enough (e.g. V of Rollnik class in dimension $d=3$). Since the asymptotic process performed in [Ca1] and leading to formulae (2.7) anyhow requires an important regularity on V, we shall not try to give optimal estimates. We simply mention the following: as mentioned in Theorem 1, for any given $D\geqslant d+1$, we have the regularity $f(t,\mathbf{n})\in C^0(\mathbb{R}_t^+;[\mathscr{S}_{2D}(\mathbb{R}^d)]^*)$ as well as $f(t,\mathbf{n})\in(L^1(\mathbb{R}_t^+;\mathscr{T}_D(\mathbb{R}^d))^*$ (see Theorem 1 for the definition of the spaces $\mathscr{T}_D(\mathbb{R}^d)$ and $\mathscr{S}_{2D}(\mathbb{R}^d)$). Using this regularity result as well as Lemma 4 given in the Appendix, it is easy to establish the following estimates, valid for any smooth test function $\phi(\mathbf{n},\mathbf{k})$ respectively $\psi(\mathbf{n})$,

$$\begin{aligned} & \left|\int_{\mathbb{R}^{2d}}\delta(\mathbf{n}^2-\mathbf{k}^2)\,T_l(\mathbf{n},\mathbf{k})\,\phi(\mathbf{n},\mathbf{k})\,d\mathbf{n}\,d\mathbf{k}\right| \\ & \qquad \leqslant C(D)^l\,\|\hat{V}\|^{l+1}_{\mathscr{S}_{2D}(\mathbb{R}^d)}\,\|\phi\|_{\mathscr{S}_D(\mathbb{R}^{2d})} \end{aligned} \quad (2.10)$$

$$\begin{aligned} & \left|\int_0^t ds\int_{\mathbb{R}^d}\mathscr{Q}_l(f)(s,\mathbf{n})\,\psi(\mathbf{n})\,d\mathbf{n}\right| \\ & \qquad \leqslant C_0C(t,D)^l\,\|\hat{V}\|^{l+1}_{\mathscr{S}_{2D}(\mathbb{R}^d)}\,\|\psi\|_{\mathscr{S}_{2D}(\mathbb{R}^d)} \end{aligned} \quad (2.11)$$

for some universal constant $C(t,D)$ depending on t and D only, and some constant C_0 depending on the norms of ρ^0 and $\hat{V}$ in (1.13). These estimates are enough to give a (weak) sense to the series defining Σ^{ld} as well as $\sum\lambda^{l+1}\mathscr{Q}_l$, for λ small enough.

(ii) Formula (2.6) is an obvious consequence of (2.5). In turn, (2.5) is derived in many textbooks, see for example [RS]. Also, note that formula

(2.7) is obviously of the form (1.15): the loss term corresponds to the contributions due to $\varepsilon_1 = \cdots = \varepsilon_l = 0$, and the gain term corresponds to all other contributions. We refer to [Ca1] for the proof of (2.7). Finally, note that (2.7) readily follows from the compact formulation (2.8)–(2.9) upon solving (2.9) iteratively to express g in terms of f, and upon writing the integral term on the right-hand-side of (2.8) under the form $\lambda \sum_{\varepsilon=0}^{1} \int_{\mathbb{R}^d} (-1)^{\varepsilon} [i\hat{V}(\mathbf{k})] \, g(\mathbf{n}-\varepsilon\mathbf{k}, \mathbf{p}+\tilde{\varepsilon}\mathbf{p}) \, d\mathbf{k}$ (see [Ca1] for details).

(iii) We refer to [Ca1] (comments after (2.2)) for other questions concerning the precise regularity and support assumptions on $\hat{V}$. Note also that formulae (2.8) and (2.9) are essentially formula (3.23)–(3.24) in [Ca1]. ∎

Having now given the explicit form of the series expansions in λ which give the values of Σ^{ld}, Σ^1, and Σ^2, we are in position to prove the identity (1.16) of our main Theorem. This is done in two steps. First we give a convenient power series expansion relating the auxiliary function g in terms of f in (2.8)–(2.9). This relies on Lemma 1. Then, we insert the value of g in equation (2.8) and turn to identifying the gain and loss terms.

2.2. Computing g in (2.9)

The two main results of this subsection are the following,

Lemma 1. The following identity holds true in the distribution sense in the variables $\mathbf{n}$, $\mathbf{m}$, $\mathbf{p}$ in $\mathbb{R}^d$, when $d \geqslant 3$,

$$\Delta(\mathbf{n}, \mathbf{p})[\Delta(\mathbf{n}, \mathbf{m}) + \Delta(\mathbf{m}, \mathbf{p})] = \Delta(\mathbf{n}, \mathbf{m}) \, \Delta(\mathbf{m}, \mathbf{p}) \tag{2.12}$$

From Lemma 4 both sides of (2.12) are defined in the distribution sense, when tested against functions of $\mathscr{S}_D(\mathbb{R}^{3d})$.

Lemma 2. Under the assumptions of Theorem 1, the auxiliary distribution g appearing in (2.9) admits the following expression,

$$g(t, \mathbf{n}, \mathbf{p}) = \sum_{l \geqslant 1} \lambda^l \Bigg[a_l(\mathbf{n}, \mathbf{p}) \, f(t, \mathbf{n}) + b_l(\mathbf{n}, \mathbf{p}) \, f(t, \mathbf{p}) + \int_{\mathbb{R}^d} c_l(\mathbf{n}, \mathbf{m}, \mathbf{p}) \, f(t, \mathbf{m}) \, d\mathbf{m} \Bigg] \tag{2.13}$$

up to defining,

$$a_l(\mathbf{n}, \mathbf{p}) = \int_{\mathbb{R}^{d(l-1)}} \Delta(\mathbf{n}, \mathbf{p}) \, \Delta(\mathbf{n}, \mathbf{k}_1) \cdots \Delta(\mathbf{n}, \mathbf{k}_{l-1}) \times \hat{V}_l(\mathbf{n}, \mathbf{k}_1, \ldots, \mathbf{k}_{l-1}, \mathbf{p}) \, d\mathbf{k}_1 \cdots d\mathbf{k}_{l-1} \tag{2.14}$$

$$b_l(\mathbf{n}, \mathbf{p}) = (-1)^l \int_{\mathbb{R}^{d(l-1)}} \Delta(\mathbf{n}, \mathbf{p})\, \Delta(\mathbf{k}_1, \mathbf{p}) \cdots \Delta(\mathbf{k}_{l-1}, \mathbf{p}) \\ \times \hat{V}_l(\mathbf{n}, \mathbf{k}_1, \ldots, \mathbf{k}_{l-1}, \mathbf{p})\, d\mathbf{k}_1 \cdots d\mathbf{k}_{l-1} \tag{2.15}$$

$$c_l(\mathbf{n}, \mathbf{m}, \mathbf{p}) = \sum_{s=0}^{l-2} (-1)^{s+1} \int_{\mathbb{R}^{d(l-2)}} \Delta(\mathbf{n}, \mathbf{m})\, \Delta(\mathbf{k}_1, \mathbf{m}) \cdots \Delta(\mathbf{k}_s, \mathbf{m}) \\ \times \Delta(\mathbf{m}, \mathbf{k}_{s+1}) \cdots \Delta(\mathbf{m}, \mathbf{k}_{l-2})\, \Delta(\mathbf{m}, \mathbf{p}) \\ \times \hat{V}_l(\mathbf{n}, \mathbf{k}_1, \ldots, \mathbf{k}_s, \mathbf{m}, \mathbf{k}_{s+1}, \ldots, \mathbf{k}_{l-2}, \mathbf{p})\, d\mathbf{k}_1 \cdots d\mathbf{k}_{l-2} \tag{2.16}$$

Remark 3. From Lemma 4 and the regularity of the distribution f, the distributions appearing above are well defined: more precisely, for any test function $\psi(\mathbf{n}, \mathbf{p})$ the following is easily established,

$$\left| \int_0^t ds \int_{\mathbb{R}^{2d}} g(s, \mathbf{n}, \mathbf{p})\, \psi(\mathbf{n}, \mathbf{p})\, d\mathbf{n}\, d\mathbf{p} \right| \leqslant \sum_{l \geqslant 1} |\lambda|^l\, \|\hat{V}\|^l_{\mathscr{S}_{2D}(\mathbb{R}^d)}\, \|\psi(\mathbf{n}, \mathbf{p})\|_{\mathscr{S}_{2D}(\mathbb{R}^{2d})} \quad \blacksquare \tag{2.17}$$

Proof of Lemma 1. The proof relies on the following observation: from Lemma 4 the left-hand-side of (2.12) is,

$$= \lim_{\alpha \to 0} \frac{i}{\mathbf{n}^2 - \mathbf{p}^2 + i\alpha} \left[\frac{i}{\mathbf{n}^2 - \mathbf{m}^2 + i\alpha} + \frac{i}{\mathbf{m}^2 - \mathbf{p}^2 + i\alpha} \right]$$

hence,

$$\begin{aligned}
&= \lim_{\alpha \to 0} \frac{i}{\mathbf{n}^2 - \mathbf{p}^2 + i\alpha} \frac{i(\mathbf{n}^2 - \mathbf{p}^2 + 2i\alpha)}{(\mathbf{n}^2 - \mathbf{m}^2 + i\alpha)(\mathbf{m}^2 - \mathbf{p}^2 + i\alpha)} \\
&= \lim_{\alpha \to 0} \left(\frac{-1}{(\mathbf{n}^2 - \mathbf{m}^2 + i\alpha)(\mathbf{m}^2 - \mathbf{p}^2 + i\alpha)} \right. \\
&\qquad \left. - \frac{i\alpha}{(\mathbf{n}^2 - \mathbf{p}^2 + i\alpha)(\mathbf{n}^2 - \mathbf{m}^2 + i\alpha)(\mathbf{m}^2 - \mathbf{p}^2 + i\alpha)} \right) \\
&= \lim_{\alpha \to 0} (\Delta_\alpha(\mathbf{n}, \mathbf{m})\, \Delta_\alpha(\mathbf{m}, \mathbf{p}) + \alpha\, \Delta_\alpha(\mathbf{n}, \mathbf{p})\, \Delta_\alpha(\mathbf{n}, \mathbf{m})\, \Delta_\alpha(\mathbf{m}, \mathbf{p})) \\
&= \Delta(\mathbf{n}, \mathbf{m})\, \Delta(\mathbf{m}, \mathbf{p}) \quad \blacksquare
\end{aligned}$$

Proof of Lemma 2. As mentioned in Remark 3, the fact that all distributions appearing in Lemma 2 are well-defined, and the series in (2.13) indeed converges, is easily proved by means of Lemma 4.

Now, the proof of formulae (2.14)–(2.16) is obtained by induction.

Upon solving (2.9) iteratively, it is obvious that g admits a power expansion (in λ) of the form (2.13). Also, we readily obtain the lower order term of this expansion,

$$g(t, \mathbf{n}, \mathbf{p}) = -i\lambda \Delta(\mathbf{n}, \mathbf{p})\, \hat{V}(\mathbf{n}-\mathbf{p})[f(t, \mathbf{p}) - f(t, \mathbf{n})] + O(\lambda^2)$$

This gives the first terms of the expansion (2.13), namely,

$$a_1(\mathbf{n}, \mathbf{p}) = +i\Delta(\mathbf{n}, \mathbf{p})\, \hat{V}(\mathbf{n}-\mathbf{p})$$

$$b_1(\mathbf{n}, \mathbf{p}) = -i\Delta(\mathbf{n}, \mathbf{p})\, \hat{V}(\mathbf{n}-\mathbf{p})$$

$$c_1(\mathbf{n}, \mathbf{m}, \mathbf{p}) = 0$$

as claimed in (2.14)–(2.16).

We compute the next coefficients by induction. Assume that a_l, b_l and c_l are indeed given by (2.14)–(2.16). Then, using Eq. (2.9), we easily obtain that the next coefficients a_{l+1}, b_{l+1}, and c_{l+1} are given by,

$$a_{l+1}(\mathbf{n}, \mathbf{p}) = i\Delta(\mathbf{n}, \mathbf{p}) \int_{\mathbb{R}^d} \hat{V}(\mathbf{k}-\mathbf{p})\, a_l(\mathbf{n}, \mathbf{k})\, d\mathbf{k} \tag{2.18}$$

$$b_{l+1}(\mathbf{n}, \mathbf{p}) = -i\Delta(\mathbf{n}, \mathbf{p}) \int_{\mathbb{R}^d} \hat{V}(\mathbf{n}-\mathbf{k})\, b_l(\mathbf{k}, \mathbf{p})\, d\mathbf{k} \tag{2.19}$$

$$\begin{aligned} c_{l+1}(\mathbf{n}, \mathbf{m}, \mathbf{p}) = -i\Delta(\mathbf{n}, \mathbf{p}) \bigg[&\hat{V}(\mathbf{n}-\mathbf{m})\, a_l(\mathbf{m}, \mathbf{p}) - \hat{V}(\mathbf{m}-\mathbf{p})\, b_l(\mathbf{n}, \mathbf{m}) \\ &+ \int_{\mathbb{R}^d} [\hat{V}(\mathbf{n}-\mathbf{k})\, c_l(\mathbf{k}, \mathbf{m}, \mathbf{p}) - \hat{V}(\mathbf{k}-\mathbf{p})\, c_l(\mathbf{n}, \mathbf{m}, \mathbf{k})]\, d\mathbf{k} \bigg] \end{aligned} \tag{2.20}$$

Clearly, (2.18) together with the value of a_l give the correct value of a_{l+1} given by (2.14), and the same holds for b_{l+1}. There remains to compute c_{l+1}. For that purpose, we insert the values of a_l, b_l and c_l in (2.20) and obtain,

$$\begin{aligned} c_{l+1}(\mathbf{n}, \mathbf{m}, \mathbf{p}) = -\Delta(\mathbf{n}, \mathbf{p}) \bigg[&\int_{\mathbb{R}^{(l-1)d}} \hat{V}_1(\mathbf{n}, \mathbf{m})\, \Delta(\mathbf{m}, \mathbf{p})\, \Delta(\mathbf{m}, \mathbf{k}_1) \cdots \Delta(\mathbf{m}, \mathbf{k}_{l-1}) \\ &\times \hat{V}_l(\mathbf{m}, \mathbf{k}_1, \ldots, \mathbf{k}_{l-1}, \mathbf{p})\, d\mathbf{k}_1 \cdots d\mathbf{k}_{l-1} \\ &-(-1)^l \int_{\mathbb{R}^{(l-1)d}} \hat{V}_1(\mathbf{m}, \mathbf{p})\, \Delta(\mathbf{n}, \mathbf{m})\, \Delta(\mathbf{k}_1, \mathbf{m}) \cdots \Delta(\mathbf{k}_{l-1}, \mathbf{m}) \\ &\times \hat{V}_l(\mathbf{n}, \mathbf{k}_1, \ldots, \mathbf{k}_{l-1}, \mathbf{m})\, d\mathbf{k}_1 \cdots d\mathbf{k}_{l-1} \end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^{(l-1)d}} \hat{V}_1(\mathbf{n},\mathbf{k}) \sum_{s=0}^{l-2} (-1)^{s+1}\, \Delta(\mathbf{k},\mathbf{m})\, \Delta(\mathbf{k}_1,\mathbf{m}) \cdots \Delta(\mathbf{k}_s,\mathbf{m}) \\
&\times \Delta(\mathbf{m},\mathbf{k}_{s+1}) \cdots \Delta(\mathbf{m},\mathbf{k}_{l-2})\, \Delta(\mathbf{m},\mathbf{p}) \\
&\times \hat{V}_l(\mathbf{k},\mathbf{k}_1,\ldots,\mathbf{k}_s,\mathbf{m},\mathbf{k}_{s+1},\ldots,\mathbf{k}_{l-2},\mathbf{p})\, d\mathbf{k}\, d\mathbf{k}_1 \cdots d\mathbf{k}_{l-2} \\
&-\int_{\mathbb{R}^{(l-1)d}} \hat{V}_1(\mathbf{k},\mathbf{p}) \sum_{s=0}^{l-2} (-1)^{s+1}\, \Delta(\mathbf{n},\mathbf{m})\, \Delta(\mathbf{k}_1,\mathbf{m}) \cdots \Delta(\mathbf{k}_s,\mathbf{m}) \\
&\times \Delta(\mathbf{m},\mathbf{k}_{s+1}) \cdots \Delta(\mathbf{m},\mathbf{k}_{l-2})\, \Delta(\mathbf{m},\mathbf{k}) \\
&\times \hat{V}_l(\mathbf{n},\mathbf{k}_1,\ldots,\mathbf{k}_s,\mathbf{m},\mathbf{k}_{s+1},\ldots,\mathbf{k}_{l-2},\mathbf{k})\, d\mathbf{k}\, d\mathbf{k}_1 \cdots d\mathbf{k}_{l-2} \Bigg]
\end{aligned}
$$

Now we treat separately the case $s=0$ in the second sum over s, and set $\mathbf{k}=\mathbf{k}_{l-1}$ in the corresponding integral. Also, we treat separately the case $s=l-2$ in the first sum over s, and make the change of variables $\mathbf{k}\to\mathbf{k}_1$, $\mathbf{k}_1\to\mathbf{k}_2,\ldots,\mathbf{k}_{l-2}\to\mathbf{k}_{l-1}$ in the corresponding integral. The remaining terms in the two sums over s are treated as follows: in the second sum, we simply change variables $\mathbf{k}\to\mathbf{k}_{l-1}$ in the integral term; in the first sum we set $s\to s-1$, and change variables $\mathbf{k}\to\mathbf{k}_1$, $\mathbf{k}_1\to\mathbf{k}_2,\ldots,\mathbf{k}_{l-2}\to\mathbf{k}_{l-1}$ in the integral term. All these operations give,

$$
\begin{aligned}
c_{l+1}(\mathbf{n},\mathbf{m},\mathbf{p}) = \int_{\mathbb{R}^{(l-1)d}} d\mathbf{k}_1 \cdots d\mathbf{k}_{l-1} \Bigg(&-\hat{V}_{l+1}(\mathbf{n},\mathbf{m},\mathbf{k}_1,\ldots,\mathbf{k}_{l-1},\mathbf{p}) \\
&\times \Delta(\mathbf{n},\mathbf{p})[\Delta(\mathbf{m},\mathbf{p})\, \Delta(\mathbf{m},\mathbf{k}_1) \cdots \Delta(\mathbf{m},\mathbf{k}_{l-1}) \\
&+\Delta(\mathbf{n},\mathbf{m})\, \Delta(\mathbf{m},\mathbf{k}_1) \cdots \Delta(\mathbf{m},\mathbf{k}_{l-1})] \\
&-(-1)^{l+1} \hat{V}_{l+1}(\mathbf{n},\mathbf{k}_1,\ldots,\mathbf{k}_{l-1},\mathbf{m},\mathbf{p})\, \Delta(\mathbf{n},\mathbf{p}) \\
&\times [\Delta(\mathbf{n},\mathbf{m})\, \Delta(\mathbf{k}_1,\mathbf{m}) \cdots \Delta(\mathbf{k}_{l-1},\mathbf{m}) \\
&+\Delta(\mathbf{m},\mathbf{p})\, \Delta(\mathbf{k}_1,\mathbf{m}) \cdots \Delta(\mathbf{k}_{l-1},\mathbf{m})] \\
&-\sum_{s=1}^{l-2} (-1)^s\, \hat{V}_{l+1}(\mathbf{n},\mathbf{k}_1,\ldots,\mathbf{k}_s,\mathbf{m},\mathbf{k}_{s+1},\ldots,\mathbf{k}_{l-1},\mathbf{p})\, \Delta(\mathbf{n},\mathbf{p}) \\
&\times [\Delta(\mathbf{m},\mathbf{p})\, \Delta(\mathbf{k}_1,\mathbf{m}) \cdots \Delta(\mathbf{k}_s,\mathbf{m})\, \Delta(\mathbf{m},\mathbf{k}_{s+1}) \cdots \Delta(\mathbf{m},\mathbf{k}_{l-1}) \\
&+\Delta(\mathbf{n},\mathbf{m})\, \Delta(\mathbf{k}_1,\mathbf{m}) \cdots \Delta(\mathbf{k}_s,\mathbf{m})\, \Delta(\mathbf{m},\mathbf{k}_{s+1}) \cdots \Delta(\mathbf{m},\mathbf{k}_{l-1})] \Bigg)
\end{aligned}
$$

Hence, using Lemma 1 to simplify the above sums of $\varDelta$-distributions, we obtain,

$$
\begin{aligned}
c_{l+1}(\mathbf{n}, \mathbf{m}, \mathbf{p}) = \int_{\mathbb{R}^{(l-1)d}} d\mathbf{k}_1 \cdots d\mathbf{k}_{l-1} \Bigg(& -\hat{V}_{l+1}(\mathbf{n}, \mathbf{m}, \mathbf{k}_1, \ldots, \mathbf{k}_{l-1}, \mathbf{p}) \\
& \times \varDelta(\mathbf{n}, \mathbf{m})\, \varDelta(\mathbf{m}, \mathbf{k}_1) \cdots \varDelta(\mathbf{m}, \mathbf{k}_{l-1})\, \varDelta(\mathbf{m}, \mathbf{p}) \\
& -(-1)^{l+1} \hat{V}_{l+1}(\mathbf{n}, \mathbf{k}_1, \ldots, \mathbf{k}_{l-1}, \mathbf{m}, \mathbf{p}) \\
& \times \varDelta(\mathbf{n}, \mathbf{m})\, \varDelta(\mathbf{k}_1, \mathbf{m}) \cdots \varDelta(\mathbf{k}_{l-1}, \mathbf{m})\, \varDelta(\mathbf{m}, \mathbf{p}) \\
& - \sum_{s=1}^{l-2} (-1)^s\, \hat{V}_{l+1}(\mathbf{n}, \mathbf{k}_1, \ldots, \mathbf{k}_s, \mathbf{m}, \mathbf{k}_{s+1}, \ldots, \mathbf{k}_{l-1}, \mathbf{p}) \\
& \times \varDelta(\mathbf{n}, \mathbf{m})\, \varDelta(\mathbf{k}_1, \mathbf{m}) \cdots \varDelta(\mathbf{k}_s, \mathbf{m}) \\
& \times \varDelta(\mathbf{m}, \mathbf{k}_{s+1}) \cdots \varDelta(\mathbf{m}, \mathbf{k}_{l-1})\, \varDelta(\mathbf{m}, \mathbf{p}) \Bigg)
\end{aligned}
$$

This ends the proof of Lemma 2. ▮

It is now an easy task to prove the main Theorem, as we do in the next section.

2.3. Proof of the main Theorem

To prove our main Theorem, we establish the following,

Lemma 3. Under the assumptions of Theorem 1, we have,

(i) The cross-section Σ^1 in Eq. (1.15) satisfies,

$$
\begin{aligned}
\Sigma^1(\mathbf{n}, \mathbf{k}) &= \sum_{l \geqslant 1} \lambda^{l+1} \Sigma_l^1(\mathbf{n}, \mathbf{k}) \\
\Sigma_l^1(\mathbf{n}, \mathbf{k}) &:= 2\pi\delta(\mathbf{n}^2 - \mathbf{k}^2) \sum_{s=0}^{l-1} \int_{\mathbb{R}^{(l-1)d}} (-1)^{s+1} \\
& \quad \times \hat{V}_{l+1}(\mathbf{n}, \mathbf{k}_1, \ldots, \mathbf{k}_s, \mathbf{k}, \mathbf{k}_{s+1}, \ldots, \mathbf{k}_{l-1}, \mathbf{n}) \\
& \quad \times \varDelta(\mathbf{k}_1, \mathbf{n}) \cdots \varDelta(\mathbf{k}_s, \mathbf{n})\, \varDelta(\mathbf{n}, \mathbf{k}_{s+1}) \cdots \varDelta(\mathbf{n}, \mathbf{k}_{l-1})\, d\mathbf{k}_1 \cdots d\mathbf{k}_{l-1}
\end{aligned}
$$

(ii) In particular, we have,

$$\Sigma^1(\mathbf{n}, \mathbf{k}) = \Sigma^{\mathrm{ld}}(\mathbf{n}, \mathbf{k}) = 2\pi\delta(\mathbf{n}^2 - \mathbf{k}^2) |T(\mathbf{k}, \mathbf{n})|^2 \tag{2.21}$$

(iii) The cross-section Σ^2 in Eq. (1.15) satisfies,

$$\int_{\mathbb{R}^d} \Sigma^2(\mathbf{n}, \mathbf{k})\, d\mathbf{k} = \int_{\mathbb{R}^d} \left[\sum_{l \geqslant 1} \lambda^{l+1} \Sigma_l^2(\mathbf{n}, \mathbf{k}) \right] d\mathbf{k} \tag{2.22}$$

$$\begin{aligned} \int_{\mathbb{R}^d} \Sigma_l^2(\mathbf{n}, \mathbf{k})\, d\mathbf{k} := \int_{\mathbb{R}^{ld}} & \hat{V}_{l+1}(\mathbf{n}, \mathbf{k}_1,\ldots, \mathbf{k}_l, \mathbf{n})[(-1)^l \Delta(\mathbf{k}_1, \mathbf{n}) \cdots \Delta(\mathbf{k}_l, \mathbf{n}) \\ & - \Delta(\mathbf{n}, \mathbf{k}_1) \cdots \Delta(\mathbf{n}, \mathbf{k}_{l-1})]\, d\mathbf{k}_1 \cdots d\mathbf{k}_l \end{aligned} \tag{2.23}$$

(iv) In particular, we have,

$$\int_{\mathbb{R}^d} \Sigma^2(\mathbf{n}, \mathbf{k})\, d\mathbf{k} = 2\pi \int_{\mathbb{R}^d} \delta(\mathbf{n}^2 - \mathbf{k}^2)\, |T(\mathbf{n}, \mathbf{k})|^2\, d\mathbf{k} \tag{2.24}$$

Remark 4. Our main theorem is a simple reformulation of parts (ii) and (iv) of the above Lemma. Note also that all the distributions and power series arising in the above Lemma are well-defined, thanks to Lemma 4. We do not write the corresponding estimates. ▮

Proof of Lemma 3. Part (ii) of the Lemma is implied by part (i), by virtue of formula (2.6). There remains to prove (i), (iii), and (iv). We first prove (i) and (iii).

To do so, we insert the series expansion (2.13) of the function g into (2.8), and identify the loss and gain terms. We write,

$$\begin{aligned} \partial_t f(t, \mathbf{n}) &= \lambda \int_{\mathbb{R}^d} [\hat{V}(\mathbf{n}, \mathbf{k})\, g(t, \mathbf{k}, \mathbf{n}) - \hat{V}(\mathbf{k}, \mathbf{n})\, g(t, \mathbf{n}, \mathbf{k})]\, d\mathbf{k} \\ &= \sum_{l \geqslant 1} \lambda^{l+1} d_l(t, \mathbf{n}) \end{aligned} \tag{2.25}$$

up to introducing,

$$\begin{aligned} d_l(t, \mathbf{n}) = - \int_{\mathbb{R}^{ld}} & \hat{V}_1(\mathbf{n}, \mathbf{k})[\Delta(\mathbf{k}, \mathbf{n})\, \Delta(\mathbf{k}, \mathbf{k}_1) \cdots \Delta(\mathbf{k}, \mathbf{k}_{l-1}) \\ & \times \hat{V}_l(\mathbf{k}, \mathbf{k}_1,\ldots, \mathbf{k}_{l-1}, \mathbf{n})\, f(t, \mathbf{k}) \\ & + (-1)^l \Delta(\mathbf{k}, \mathbf{n})\, \Delta(\mathbf{k}_1, \mathbf{n}) \cdots \Delta(\mathbf{k}_{l-1}, \mathbf{n}) \\ & \times \hat{V}_l(\mathbf{k}, \mathbf{k}_1,\ldots, \mathbf{k}_{l-1}, \mathbf{n})\, f(t, \mathbf{n})]\, d\mathbf{k}\, d\mathbf{k}_1 \cdots d\mathbf{k}_{l-1} \\ + \int_{\mathbb{R}^{ld}} & \hat{V}_1(\mathbf{k}, \mathbf{n})[\Delta(\mathbf{n}, \mathbf{k})\, \Delta(\mathbf{n}, \mathbf{k}_1) \cdots \Delta(\mathbf{n}, \mathbf{k}_{l-1}) \\ & \times \hat{V}_l(\mathbf{n}, \mathbf{k}_1,\ldots, \mathbf{k}_{l-1}, \mathbf{k})\, f(t, \mathbf{n}) \end{aligned}$$

$$
\begin{aligned}
&+(-1)^l \Delta(\mathbf{n},\mathbf{k})\,\Delta(\mathbf{k}_1,\mathbf{k})\cdots\Delta(\mathbf{k}_{l-1},\mathbf{k})\\
&\times \hat{V}_l(\mathbf{n},\mathbf{k}_1,\ldots,\mathbf{k}_{l-1},\mathbf{k})\, f(t,\mathbf{k})]\, d\mathbf{k}\, d\mathbf{k}_1\cdots d\mathbf{k}_{l-1}\\
&+\sum_{s=0}^{l-2}\int_{\mathbb{R}^{ld}} f(t,\mathbf{m})[(-1)^s\,\hat{V}_1(\mathbf{n},\mathbf{k})\\
&\times \hat{V}_l(\mathbf{k},\mathbf{k}_1,\ldots,\mathbf{k}_s,\mathbf{m},\mathbf{k}_{s+1},\ldots,\mathbf{k}_{l-2},\mathbf{n})\\
&\times \Delta(\mathbf{k},\mathbf{m})\,\Delta(\mathbf{k}_1,\mathbf{m})\cdots\Delta(\mathbf{k}_s,\mathbf{m})\,\Delta(\mathbf{m},\mathbf{k}_{s+1})\cdots\Delta(\mathbf{m},\mathbf{k}_{l-2})\Delta(\mathbf{m},\mathbf{n})\\
&-(-1)^s\,\hat{V}_1(\mathbf{k},\mathbf{n})\,\hat{V}_l(\mathbf{n},\mathbf{k}_1,\ldots,\mathbf{k}_s,\mathbf{m},\mathbf{k}_{s+1},\ldots,\mathbf{k}_{l-2},\mathbf{k})\\
&\times \Delta(\mathbf{n},\mathbf{m})\,\Delta(\mathbf{k}_1,\mathbf{m})\cdots\Delta(\mathbf{k}_s,\mathbf{m})\,\Delta(\mathbf{m},\mathbf{k}_{s+1})\cdots\Delta(\mathbf{m},\mathbf{k}_{l-2})\,\Delta(\mathbf{m},\mathbf{k})]\\
&\times d\mathbf{m}\, d\mathbf{k}\, d\mathbf{k}_1\cdots d\mathbf{k}_{l-2}
\end{aligned}
$$

Hence we recover, upon renaming the integration variables,

$$
\begin{aligned}
d_l(t,\mathbf{n}) = {}& f(t,\mathbf{n})\int_{\mathbb{R}^{ld}} \hat{V}_{l+1}(\mathbf{n},\mathbf{k}_1,\ldots,\mathbf{k}_l,\mathbf{n})\\
&\times[(-1)^{l+1}\,\Delta(\mathbf{k}_1,\mathbf{n})\cdots\Delta(\mathbf{k}_l,\mathbf{n})+\Delta(\mathbf{n},\mathbf{k}_1)\cdots\Delta(\mathbf{n},\mathbf{k}_l)]\, d\mathbf{k}_1\cdots d\mathbf{k}_l\\
&+\int_{\mathbb{R}^{ld}} f(t,\mathbf{k})[-\hat{V}_{l+1}(\mathbf{n},\mathbf{k},\mathbf{k}_1,\ldots,\mathbf{k}_{l-1},\mathbf{n})\\
&\times \Delta(\mathbf{k},\mathbf{n})\,\Delta(\mathbf{k},\mathbf{k}_1)\cdots\Delta(\mathbf{k},\mathbf{k}_{l-1})\\
&+(-1)^l\,\hat{V}_{l+1}(\mathbf{n},\mathbf{k}_1,\ldots,\mathbf{k}_{l-1},\mathbf{k},\mathbf{n})\,\Delta(\mathbf{n},\mathbf{k})\,\Delta(\mathbf{k}_1,\mathbf{k})\cdots\Delta(\mathbf{k}_{l-1},\mathbf{k})]\\
&\times d\mathbf{k}\, d\mathbf{k}_1\cdots d\mathbf{k}_{l-1}\\
&+\sum_{s=0}^{l-2}\int_{\mathbb{R}^{ld}} f(t,\mathbf{k})(-1)^s\,\hat{V}_{l+1}(\mathbf{n},\mathbf{m},\mathbf{k}_1,\ldots,\mathbf{k}_s,\mathbf{k},\mathbf{k}_{s+1},\ldots,\mathbf{k}_{l-2},\mathbf{n})\\
&\times \Delta(\mathbf{m},\mathbf{k})\,\Delta(\mathbf{k}_1,\mathbf{k})\cdots\Delta(\mathbf{k}_s,\mathbf{k})\,\Delta(\mathbf{k},\mathbf{k}_{s+1})\cdots\Delta(\mathbf{k},\mathbf{k}_{l-2})\,\Delta(\mathbf{k},\mathbf{n})\\
&\times d\mathbf{m}\, d\mathbf{k}\, d\mathbf{k}_1\cdots d\mathbf{k}_{l-2}\\
&-\sum_{s=0}^{l-2}\int_{\mathbb{R}^{ld}} f(t,\mathbf{k})(-1)^s\,\hat{V}_{l+1}(\mathbf{n},\mathbf{k}_1,\ldots,\mathbf{k}_s,\mathbf{k},\mathbf{k}_{s+1},\ldots,\mathbf{k}_{l-2},\mathbf{m},\mathbf{n})\\
&\times \Delta(\mathbf{n},\mathbf{k})\,\Delta(\mathbf{k}_1,\mathbf{k})\cdots\Delta(\mathbf{k}_s,\mathbf{k})\,\Delta(\mathbf{k},\mathbf{k}_{s+1})\cdots\Delta(\mathbf{k},\mathbf{k}_{l-2})\,\Delta(\mathbf{k},\mathbf{m})\\
&\times d\mathbf{m}\, d\mathbf{k}\, d\mathbf{k}_1\cdots d\mathbf{k}_{l-2}
\end{aligned}
\tag{2.26}
$$

Clearly, formulae (2.25) and (2.26) establish formulae (2.22) and (2.23) for the loss term in (1.15), so that (iii) is proved. There remains to identify the gain term. From (2.26), we obtain,

$$\begin{aligned}\Sigma_l^1(\mathbf{n},\mathbf{k}) = \int_{\mathbb{R}^{(l-1)d}} \Bigg[& -\hat{V}_{l+1}(\mathbf{n},\mathbf{k},\mathbf{k}_1,\ldots,\mathbf{k}_{l-1},\mathbf{n})\,\Delta(\mathbf{k},\mathbf{n})\,\Delta(\mathbf{k},\mathbf{k}_1)\cdots\Delta(\mathbf{k},\mathbf{k}_{l-1}) \\ & +(-1)^l\,\hat{V}_{l+1}(\mathbf{n},\mathbf{k}_1,\ldots,\mathbf{k}_{l-1},\mathbf{k},\mathbf{n})\,\Delta(\mathbf{n},\mathbf{k})\,\Delta(\mathbf{k}_1,\mathbf{k})\cdots\Delta(\mathbf{k}_{l-1},\mathbf{k}) \\ & +\sum_{s=0}^{l-2}(-1)^s\,\hat{V}_{l+1}(\mathbf{n},\mathbf{k}_1,\ldots,\mathbf{k}_{s+1},\mathbf{k},\mathbf{k}_{s+2},\ldots,\mathbf{k}_{l-1},\mathbf{n}) \\ & \times\Delta(\mathbf{k}_1,\mathbf{k})\cdots\Delta(\mathbf{k}_{s+1},\mathbf{k})\,\Delta(\mathbf{k},\mathbf{k}_{s+2})\cdots\Delta(\mathbf{k},\mathbf{k}_{l-1})\,\Delta(\mathbf{k},\mathbf{n}) \\ & -\sum_{s=0}^{l-2}(-1)^s\,\hat{V}_{l+1}(\mathbf{n},\mathbf{k}_1,\ldots,\mathbf{k}_s,\mathbf{k},\mathbf{k}_{s+1},\ldots,\mathbf{k}_{l-1},\mathbf{n}) \\ & \times\Delta(\mathbf{n},\mathbf{k})\,\Delta(\mathbf{k}_1,\mathbf{k})\cdots\Delta(\mathbf{k}_s,\mathbf{k})\,\Delta(\mathbf{k},\mathbf{k}_{s+1})\cdots\Delta(\mathbf{k},\mathbf{k}_{l-1})\Bigg] \\ & \times d\mathbf{k}_1\cdots d\mathbf{k}_{l-1}\end{aligned}$$

We treat separately the case $s=0$ in the second sum over s, as well as the term $s=l-2$ in the first sum. Upon reindexing some variables, we obtain,

$$\begin{aligned}\Sigma_l^1(\mathbf{n},\mathbf{k}) = \int_{\mathbb{R}^{(l-1)d}} \Bigg[& -\hat{V}_{l+1}(\mathbf{n},\mathbf{k},\mathbf{k}_1,\ldots,\mathbf{k}_{l-1},\mathbf{n}) \\ & \times[\Delta(\mathbf{k},\mathbf{n})+\Delta(\mathbf{n},\mathbf{k})]\,\Delta(\mathbf{k},\mathbf{k}_1)\cdots\Delta(\mathbf{k},\mathbf{k}_{l-1}) \\ & +(-1)^l\,\hat{V}_{l+1}(\mathbf{n},\mathbf{k}_1,\ldots,\mathbf{k}_{l-1},\mathbf{k},\mathbf{n}) \\ & \times[\Delta(\mathbf{k},\mathbf{n})+\Delta(\mathbf{n},\mathbf{k})]\,\Delta(\mathbf{k}_1,\mathbf{k})\cdots\Delta(\mathbf{k}_{l-1},\mathbf{k}) \\ & +\sum_{s=1}^{l-2}(-1)^{s+1}\,\hat{V}_{l+1}(\mathbf{n},\mathbf{k}_1,\ldots,\mathbf{k}_s,\mathbf{k},\mathbf{k}_{s+1},\ldots,\mathbf{k}_{l-1},\mathbf{n}) \\ & \times[\Delta(\mathbf{k},\mathbf{n})+\Delta(\mathbf{n},\mathbf{k})]\,\Delta(\mathbf{k}_1,\mathbf{k})\cdots\Delta(\mathbf{k}_s,\mathbf{k}) \\ & \times\Delta(\mathbf{k},\mathbf{k}_{s+1})\cdots\Delta(\mathbf{k},\mathbf{k}_{l-1})\Bigg]\,d\mathbf{k}_1\cdots d\mathbf{k}_{l-1}\end{aligned}$$

Observing that (see (2.4)),

$$\Delta(\mathbf{n},\mathbf{k})+\Delta(\mathbf{k},\mathbf{n}) = 2\pi\delta(\mathbf{n}^2-\mathbf{k}^2)$$

we obtain part (i) of the Lemma. This ends the proof of (i) and (iii).

To prove (iv), we take a smooth test function $\tilde{f}(\mathbf{n}) \in C_c^\infty(\mathbb{R}^d)$. We want to evaluate the quantity,

$$A := \int_{\mathbb{R}^{2d}} [\Sigma^1(\mathbf{n}, \mathbf{k})\, \tilde{f}(\mathbf{k}) - \Sigma^2(\mathbf{n}, \mathbf{k})\, \tilde{f}(\mathbf{n})]\, d\mathbf{n}\, d\mathbf{k}$$

To do so, we use the equivalence between formulae (2.8)–(2.9) and (1.15) (or equivalently (2.7)). Thus, to $\tilde{f}$ we associate $\tilde{g}(\mathbf{n}, \mathbf{p})$ defined through formula (2.9) with $f(t, .)$ now replaced by $\tilde{f}(.)$. We readily have,

$$A = -i\lambda \int_{\mathbb{R}^{2d}} [\hat{V}(\mathbf{n}-\mathbf{k})\, \tilde{g}(\mathbf{k}, \mathbf{n}) - \hat{V}(\mathbf{k}-\mathbf{n})\, \tilde{g}(\mathbf{n}, \mathbf{k})]\, d\mathbf{n}\, d\mathbf{k} \tag{2.27}$$

Now, by virtue of Lemma 2 applied to $\tilde{f}$ and $\tilde{g}$, we obtain,

$$\tilde{g}(\mathbf{n}, \mathbf{p}) = \sum_{l \geqslant 1} \lambda^l \left[a_l(\mathbf{n}, \mathbf{p})\, \tilde{f}(\mathbf{n}) + b_l(\mathbf{n}, \mathbf{p})\, \tilde{f}(\mathbf{p}) + \int_{\mathbb{R}^d} c_l(\mathbf{n}, \mathbf{m}, \mathbf{p})\, \tilde{f}(\mathbf{m})\, d\mathbf{m} \right] \tag{2.28}$$

where the coefficients a_l, b_l, c_l are given by formulae (2.14) through (2.16). Lemma 4 now asserts that the change of variables $(\mathbf{n}, \mathbf{k}) \to (\mathbf{k}, \mathbf{n})$ is allowed in (2.27), and we recover,

$$A = -i\lambda \int_{\mathbb{R}^{2d}} [\hat{V}(\mathbf{n}-\mathbf{k})\, \tilde{g}(\mathbf{k}, \mathbf{n}) - \hat{V}(\mathbf{k}-\mathbf{n})\, \tilde{g}(\mathbf{n}, \mathbf{k})]\, d\mathbf{n}\, d\mathbf{k} = 0 \tag{2.29}$$

Since (2.29) is valid for any smooth $\tilde{f}$, we obtain,

$$\int_{\mathbb{R}^d} \Sigma^1(\mathbf{k}, \mathbf{n})\, d\mathbf{k} = \int_{\mathbb{R}^d} \Sigma^2(\mathbf{n}, \mathbf{k})\, d\mathbf{k} \tag{2.30}$$

as distributions in the variable $\mathbf{n}$. This together with part (ii) of Lemma 3 establishes (iv). ∎

APPENDIX

Lemma 4. Let $l \in \mathbb{N}$, $l \geqslant 1$. Let $\mathbf{m}, \mathbf{k}_1, \ldots, \mathbf{k}_l$ be variables in $\mathbb{R}^d$. Let $0 \leqslant s \leqslant l$. Assume $d \geqslant 3$. Then,

(i) the following distribution is well defined over $\mathbb{R}^{(l+1)\,d}$,

$$\Delta(\mathbf{k}_1, \mathbf{m}) \cdots \Delta(\mathbf{k}_s, \mathbf{m})\, \Delta(\mathbf{m}, \mathbf{k}_{s+1}) \cdots \Delta(\mathbf{m}, \mathbf{k}_l) \tag{A.1}$$

with the convention that this distribution reduces to $\Delta(\mathbf{k}_1, \mathbf{m}) \cdots \Delta(\mathbf{k}_l, \mathbf{m})$ in the case $s = l$,

resp. $\Delta(\mathbf{m}, \mathbf{k}_1) \cdots \Delta(\mathbf{m}, \mathbf{k}_l)$ in the $s = 0$. More precisely, for any $D \geqslant d+1$, there exists a constant $C(D)$ such that the duality product of the distribution (A.1) with a test function $\phi(\mathbf{m}, \mathbf{k}_1,..., \mathbf{k}_l)$ is bounded by,

$$C(D)^l \, \|\phi\|_{\mathscr{S}_D(\mathbb{R}^{(l+1)\,d})} \tag{A.2}$$

The space $\mathscr{S}_D(\mathbb{R}^{(l+1)\,d}$ is the space of functions having D moments and D derivatives in L^∞ as defined in (1.13).

(ii) For $\alpha > 0$, let,

$$\Delta_\alpha(\mathbf{n}, \mathbf{p}) := \int_0^{+\infty} \exp(i[\mathbf{n}^2 - \mathbf{p}^2]\, s - \alpha s)\, ds \tag{A.3}$$

Then, the following weak limit holds, upon testing against any test function ϕ in the space $\mathscr{S}_D(\mathbb{R}^{(l+1)\,d})$ $(D \geqslant d+1)$ appearing in (A.2),

$$\begin{aligned} &\lim_{\alpha \to 0} \Delta_\alpha(\mathbf{k}_1, \mathbf{m}) \cdots \Delta_\alpha(\mathbf{k}_s, \mathbf{m})\, \Delta_\alpha(\mathbf{m}, \mathbf{k}_{s+1}) \cdots \Delta_\alpha(\mathbf{m}, \mathbf{k}_l) \\ &\quad = \Delta(\mathbf{k}_1, \mathbf{m}) \cdots \Delta(\mathbf{k}_s, \mathbf{m})\, \Delta(\mathbf{m}, \mathbf{k}_{s+1}) \cdots \Delta(\mathbf{m}, \mathbf{k}_l) \end{aligned} \tag{A.4}$$

(iii) The statements analogous to (i) and (ii) above hold true when an additional variable $\mathbf{k}_{l+2} \in \mathbb{R}^d$ is given, and the distribution (A.1) is replaced by,

$$\begin{aligned} &\Delta(\mathbf{k}_1, \mathbf{k}_{l+2})\, \Delta(\mathbf{k}_1, \mathbf{m}) \cdots \Delta(\mathbf{k}_s, \mathbf{m})\, \Delta(\mathbf{m}, \mathbf{k}_{s+1}) \cdots \Delta(\mathbf{m}, \mathbf{k}_l) \\ \text{or,} \quad &\Delta(\mathbf{k}_{l+2}, \mathbf{k}_l)\, \Delta(\mathbf{k}_1, \mathbf{m}) \cdots \Delta(\mathbf{k}_s, \mathbf{m})\, \Delta(\mathbf{m}, \mathbf{k}_{s+1}) \cdots \Delta(\mathbf{m}, \mathbf{k}_l) \\ \text{or,} \quad &\Delta(\mathbf{k}_1, \mathbf{k}_l)\, \Delta(\mathbf{k}_1, \mathbf{m}) \cdots \Delta(\mathbf{k}_s, \mathbf{m})\, \Delta(\mathbf{m}, \mathbf{k}_{s+1}) \cdots \Delta(\mathbf{m}, \mathbf{k}_l) \end{aligned} \tag{A.5}$$

We refer to [Ca1], Lemma 3.1 and proof, for a proof of the above Lemma.

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REFERENCES

[Boh] A. Bohm, *Quantum Mechanics*, Texts and Monographs in Physics (Springer-Verlag, 1979).

[BBS] C. Boldrighini, L. A. Bunimovich, and Ya. Sinai, On the Boltzmann equation for the Lorentz gas, *J. Stat. Phys.* **32**:477–501 (1983).

[BGW] J. Bourgain, F. Golse, and B. Wennberg, On the distribution of free path lengths for the periodic Lorentz gas, *Comm. Math. Phys.* **190**:491–508 (1998).

[Boy] R. W. Boyd, *Nonlinear Optics* (Academic Press, 1992).

[Ck] D. Calecki, *Lecture* (University Paris 6, 1997).

[Ca1] F. Castella, From the von Neumann equation to the quantum Boltzmann equation in a deterministic framework, to appear in *J. Stat. Phys.* (2001).

[Ca2] F. Castella, Résultats de convergence et de non-convergence de l'équation de von Neumann périodique vers l'équation de Boltzmann quantique, Séminaire E.D.P. (Ecole Polytechnique, exposé N. XXI, 1999–2000).

[Ca3] F. Castella, On the derivation of a Quantum Boltzmann Equation from the periodic von Neumann equation, *Mod. Math. An. Num.* **33**:329–349 (1999).

[CD] F. Castella and P. Degond, Convergence de l'équation de von Neumann vers l'équation de Boltzmann Quantique dans un cadre déterministe, *C. R. Acad. Sci. Sér. I* **329**:231–236 (1999).

[CP1] F. Castella and A. Plagne, A distribution result for slices of sums of squares, To appear in *Math. Proc. Camb. Philos. Soc.*

[CP2] F. Castella, A. Plagne, Non-derivation of the Quantum Boltzmann equation from the periodic Schrödinger equation, Preprint (2001).

[CEFM] F. Castella, L. Erdös, F. Frommlet, and P. A. Markowich, Fokker–Planck equations as scaling limits of reversible quantum systems, *J. Statist. Phys.* **100**:543–601 (2000).

[CTDL] C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Mécanique Quantique, I et II, Enseignement des Sciences 16* (Hermann, 1973).

[Co] M. Combescot, *Is there a generalized Fermi Golden Rule* (Preprint Université Paris VI, 1999).

[Dü] R. Dümcke, The low density limit for an N-level system interacting with a free Bose or Fermi gas, *Comm. Math. Phys.* **97**:331–359 (1985).

[EY1] L. Erdös and H. T. Yau, Linear boltzmann equation as scaling limit of quantum lorentz gas advances, in *Differential Equations and Mathematical Physics* (Atlanta, GA, 1997), pp. 137–155; Contemp. Math., Vol. 217 (Amer. Math. Soc., Providence, RI, 1998).

[EY2] L. Erdös and H. T. Yau, Linear Boltzmann equation as the weak coupling limit of a random Schrödinger equation, *Comm. Pure Appl. Math.* **53**:667–735 (2000).

[Fi] M. V. Fischetti, Theory of electron transport in small semiconductor devices using the Pauli master equations, *J. Appl. Phys.* **83**:270–291 (1998).

[HLW] T. G. Ho, L. J. Landau, and A. J. Wilkins, On the weak coupling limit for a Fermi gas in a random potential, *Rev. Math. Phys.* **5**:209–298 (1993).

[Hö] L. Hörmander, *The Analysis of Linear Partial Differential Operators* (Springer-Verlag, Berlin, 1994).

[Hu] K. Huang, *Statistical Mechanics* (Wiley and Sons, 1963).

[KPR] J. B. Keller, G. Papanicolaou, and L. Ryzhik, Transport equations for elastic and other waves in random media, *Wave Motion* **24**:327–370 (1996).

[KL1] W. Kohn and J. M. Luttinger, *Phys. Rev.* **108**:590 (1957).

[KL2] W. Kohn and J. M. Luttinger, *Phys. Rev.* **109**:1892 (1958).

[Kr] H. J. Kreuzer, *Nonequilibrium Thermodynamics and its Statistical Foundations* (Oxford Science Publications, Monographs on Physics and Chemistry of Materials, 1983).
[Ku] R. Kubo, *J. Phys. Soc. Jap.* **12** (1958).
[La] L. J. Landau, Observation of Quantum Particles on a Large Space-Time Scale, *J. Stat. Phys.* **77**:259–309 (1994).
[Li] G. Lindblad, On the generators of Quantum dynamical semigroups, *Comm. Math. Phys.* **48**:119–130 (1976).
[Lo] R. Loudon, *The Quantum Theory of Light* (Clarendon Press, Oxford, 1991).
[MRS] P. A. Markowich, C. Ringhofer, and C. Schmeiser, *Semiconductor Equations* (Springer-Verlag, Vienna, 1990).
[NM] A. C. Newell and J. V. Moloney, Nonlinear Optics, *Advanced Topics in the Interdisciplinary Mathematical Sciences* (Addison–Wesley Publishing Company, 1992).
[Ni1] F. Nier, Asymptotic Analysis of a scaled Wigner equation and Quantum Scattering, *Transp. Theor. Stat. Phys.* **24**:591–629 (1995).
[Ni2] F. Nier, A semi-classical picture of quantum scattering, *Ann. Sci. Ec. Norm. Sup. Sér. 4* **29**:149–183 (1996).
[Pa] W. Pauli, *Festschrift zum 60. Geburtstage A. Sommerfelds* (Hirzel, Leipzig, 1928), p. 30.
[PV] F. Poupaud and A. Vasseur, *Classical and Quantum Transport in Random Media* (Preprint, University of Nice, 2001).
[Pr] I. Prigogine, *Non-Equilibrium Statistical Mechanics* (Interscience, New York, 1962).
[RS] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. III. Scattering theory* (Academic Press, New York/London, 1979).
[SSL] M. Sargent, M. O. Scully, and W. E. Lamb, *Laser Physics* (Addison–Wesley, 1977).
[Sp] H. Spohn, Derivation of the transport equation for electrons moving through random impurities, *J. Stat. Phys.* **17**:385–412 (1977).
[VH1] L. Van Hove, *Physica* **21**:517 (1955).
[VH2] L. Van Hove, *Physica* **23**:441 (1957).
[VH3] L. Van Hove, in *Fundamental Problems in Statistical Mechanics*, E. G. D. Cohen, ed. (1962), p. 157.
[Vk] N. G. Van Kampen, Stochastic processes in physics and chemistry, *Lecture Notes in Mathematics, Vol. 888* (North-Holland, 1981).
[Zw] R. Zwanzig, *Quantum Statistical Mechanics*, P. H. E. Meijer, ed. (Gordon and Breach, New York, 1966).